Discrete autoregressive model of conditional duration*

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In this paper we propose the discrete model which describes the low frequency of changes in stock prices. As a basic distribution, we used a discrete type distribution and also an autoregressive sequence. So, we named this model Discrete Autoregressive Conditional Duration (D-ACD) model. The main stochastic properties of the model are given. We apply this model on the real data set from Belgrade Stock Exchange.

Keywords: ACD-models, discrete type distribution, stopping time, estimation of parameters.

Introduction

Financial market theories are typically tested on a transaction by transaction basis. So, again the timing of these transactions can be central to understanding the economics. In this position, we looked at the price of stocks, i.e. the level of price and its changes. In this paper we will propose an alternative to fixed interval analysis.

Following [1], the arrival times of changes are treated as random variables which follow a point process and the evolution of some financial index can be represented with part by part constant function (Fig. 1). It has the constant value on the interval $[\tau_{k-1}, \tau_k)$,

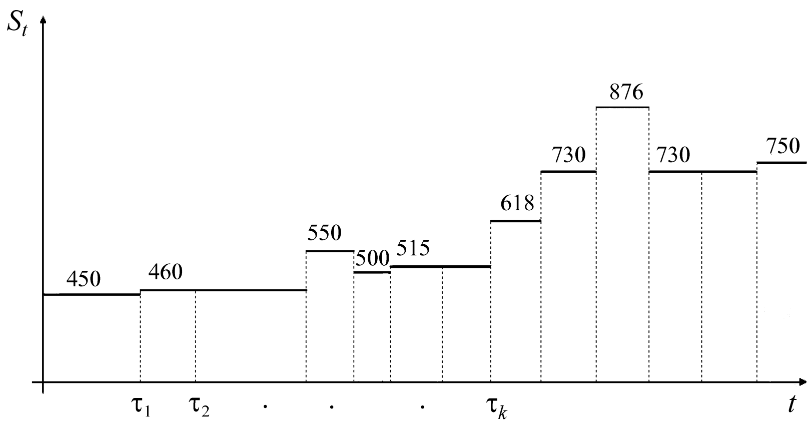

Fig. 1. Dynamics of the price of stocks of the company Hemofarm-Vršac (source: the Belgrade Stock Exchange)

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and then it changes in a random moment τ_k. The sequence $(\tau_k)_{k\geq 1}$, usually named as *stopping time sequence*, can be modeled by the autoregressive model of conditional duration (ACD model), firstly introduced by Engle and Russel [2]. They expressed irregularity in dynamics of financial indexes, considered to the stopping time sequence, by defining its members to be nonlinearly dependent of the past values. Later on, some generalizations of the basic ACD model where done by Bauwens and Giot [3] or Meitz and Teräsvirta [4].

Here, we will introduce some related model which will describe the conditional nonlinear dynamics of the stopping sequence τ_k. In Section 1, the model we propose will be described. In Section 2, we will estimate its parameters. In Section 3, Monte Carlo simulation and some application of the model will be done.

1. D-ACD model. Definition and properties

Starting from the same idea as Engle and Rassel [2] did, we define the sequence of increments

$$X_k = \tau_k - \tau_{k-1}, \quad k \geq 1,$$

where $\tau_0 \overset{a.s.}{=} 0$, and which distribution can explain the stoping time sequence τ_k. In the special case, when X_k is an i.i.d. sequence, it will be simple to investigate its proprieties. Meanwhile, there is a good reason to assume the correlation among the members of this sequence. That's why we use the model which can describe their dependance. The basic assumption of ACD model is that the sequence of increments X_k is defined as

$$X_k = \lambda_k \varepsilon_k, \quad k \geq 1, \tag{1}$$

where λ_k is an autoregressive sequence of the previous realizations of X_k, and ε_k is the sequence of independent identically distributed random variables with non-negative distribution (exponential, for instance). Now, we will introduce a new model based on the autoregressive concept.

Definition. *The sequence of increments X_k, defined by* (1)*, represents the* ***Discrete Autoregressive Conditional Duration*** (***D-ACD***) ***model*** *if it satisfies:*

(i) $(\varepsilon_k)_{k\geq 1}$ is i.i.d. sequence of random variables with the set of values $\mathbb{D} = \{0, 1, 2 \ldots\}$ and such that $E(\varepsilon_k) = 1$, $Var(\varepsilon_k) = \sigma^2$, for any $k \geq 1$, and $\varepsilon_0 \overset{a.s.}{=} 0$;

(ii) $(\lambda_k)_{k\geq 1}$ is an autoregressive sequence of random variables such that

$$\lambda_{k+1} = \lambda_k + \eta_k \, \varepsilon_k, \quad k \geq 1, \quad \lambda_0 \equiv \text{const}, \quad \lambda_0 > 0 \tag{2}$$

and

$$P\left\{ 0 < \left| \sum_{j=1}^{k} \eta_j \, \varepsilon_j \right| \leq \lambda_0 \right\} = 1, \quad k \geq 1,$$

where η_k is the i.i.d. sequence, independent of ε_k, with the uniform distribution on the set $A = \Big\{ -a, -a+1, \ldots, a-1, a \Big\}$, $a \in \mathbb{N}$, $a \leq \lambda_0$ and λ_m is independent of η_s and ε_s when $m \leq s$;

(iii) $\mathcal{F}_k = \mathcal{G}en\Big\{ (\varepsilon_j, \eta_j) \mid j = 1, 2, \ldots, k \Big\}$, for all $k \geq 1$, and $\mathcal{F}_0 = \emptyset$ represent the filtration on the probability space $(\Omega, \mathcal{F}, P)$. □

In this way, X_k is defined as the sequence of random variables with a discrete distribution which depends of distribution of the two sequences. The sequence ε_k plays the role of *white* *noise*, but with the mean value one and it has the influence on the values of X_k. From the other hand, the sequence η_k can be interpreted as *the intensity of the reaction* in the price domain and so, on the values of λ_k. As its distribution is uniform on the symmetric set A, it is equally possible for λ_k to be smaller or greater of its previous value. Because of those two distributions, it is easy to verify that

$$E(\lambda_k) = E(\lambda_{k-1}) = \cdots = E(\lambda_1) = \lambda_0(\text{const}), \quad k \geq 1,$$

and, according to (2),

$$\lambda_k \overset{a.s.}{=} \lambda_0 + \sum_{j=1}^{k-1} \eta_j \, \varepsilon_j, \quad k \geq 1.$$

The parameter a is the the upper limit of changes of the elements of the sequence η_k and, at the same time, a is the upper limit (in mean sense) for the increments X_k. When we consider the real data, a and σ^2 are unknown and we will estimate them using the known realization of increments X_k. In order to do that, further on, we will study some stochastic properties of the sequences λ_k and X_k.

According to (1) and (2), we can conclude that the random variable λ_k is $\mathcal{F}_{k-1}$ adaptive, and it follows that, for all $k \geq 1$,

$$E\big(X_k \big| \mathcal{F}_{k-1}\big) = \lambda_k, \quad Var\big(X_k \big| \mathcal{F}_{k-1}\big) = \sigma^2 \, \lambda_k^2.$$

These results are the same as are those of the basic ACD model defined by Engle and Russel. From here we have

$$E(X_k) = E(\lambda_k) = \lambda_0, \quad k \geq 1, \tag{3}$$

where λ_0 is also unknown, but it is easy to be estimated (see the next section). As,

$$Var\big(\eta_k\big) = E\big(\eta_k^2\big) = \frac{1}{3}\, a(a+1), \quad k \geq 1,$$

we have the variance of λ_k as

$$Var(\lambda_k) = \frac{k-1}{3}\, a(a+1)\,(\sigma^2+1), \quad k \geq 1.$$

Finally, it will be

$$Var(X_k) = Var(\lambda_k) + \sigma^2\, E(\lambda_k^2) = \frac{k-1}{3}\, a(a+1)(\sigma^2+1)^2 + \sigma^2\, \lambda_0^2, \quad k \geq 1.$$

In the similar way we can obtain the correlation structure of the sequences X_k and λ_k. For any $h \in \mathbb{N}$,

$$Cov(\lambda_k, \lambda_{k+h}) = Cov(\lambda_k, \lambda_{k+h-1}) = \cdots = Var(\lambda_k), \quad k \geq 1,$$

and the correlation function of λ_k is

$$Corr(\lambda_k, \lambda_{k+h}) = \sqrt{\frac{Var(\lambda_k)}{Var(\lambda_{k+h})}} = \begin{cases} \sqrt{\dfrac{k-1}{k-1+h}}, & h > 0, \\ 1, & h = 0. \end{cases}$$

On the other hand, it is easy to prove that, for any $h \in \mathbb{N}$,

$$Cov(X_k,\, X_{k+h}) = Var(\lambda_k), \quad k \geq 1$$

and the correlation function of X_k is

$$Corr(X_k, X_{k+h}) = \begin{cases} \dfrac{k-1}{\sqrt{(k-1+C)(k-1+C+h)}}, & h > 0, \\ 1, & h = 0, \end{cases}$$

where $C = \dfrac{3\sigma^2\lambda_0^2}{a(a+1)(\sigma^2+1)^2}$. The correlation functions of λ_k and X_k indicate their *nonstationarity*, but λ_k is the mean square continuous

$$\lim_{h\to 0} Corr(\lambda_k, \lambda_{k+h}) = 1.$$

Nonstationarity of X_k and λ_k is disagreeable in the practical usage of D-ACD model and it will cause the difficulties in the following estimation procedure.

2. Estimation of parameters

We need to estimate the unknown parameters of D-ACD in order to apply that model on the real data set, but we can find the only one realization $(x_1, \ldots, x_N)$ of the sequence X_k on the market. As this sequence is the non-stationary one, it is not simple to estimate three dimensional parameter $(\lambda_0,\, a,\, \sigma^2)$. Here, we will use some of the estimates which are described in Stojanoviȁ [5].

First, we will use, according to (3), the well known estimate

$$\hat{\lambda_0} = \frac{1}{N}\sum_{k=1}^{N} X_k$$

for the first component of the parameter. Then, we will consider the residual sequence (R_k) defined as $R_k = \lambda_k - \lambda_{k-1},\ k \geq 1$, which is, according to (2), the stationary sequence of random variables. The main properties of that sequence are as follow

$$E(R_k) = 0, \quad Var(R_k) = \frac{1}{3}\, a(a+1)(\sigma^2+1), \quad k \geq 1. \tag{4}$$

The values of this sequence can be estimated in the following way. We will use the conditional least square method and minimize the sum

$$Q_N(X_1, \ldots, X_N; \boldsymbol{\Lambda}) = \sum_{k=1}^{N} \Big(X_k - E(X_k \mid \mathcal{F}_{k-1}) \Big)^2$$

with respect to $\boldsymbol{\Lambda}{=}(\lambda_1, \ldots, \lambda_N)'$. It will have the minimum value when

$$\hat{\lambda}_k = X_k, \quad k = 1, \ldots, N$$

and, therefore, we can estimate the values $R_k,\ k = 1, \ldots, N$, using the sequence

$$Y_k = \hat{\lambda}_k - \hat{\lambda}_{k-1} = X_k - X_{k-1}, \quad k = 1, \ldots, N,$$

where $X_0 \overset{as}{=} 0$. Obviously, the sequence (Y_k) is a nonstationary one, with $E(Y_k) = 0$, but the following convergence is valid

$$\frac{1}{N^2}\sum_{k=1}^{N} Var(Y_k) \longrightarrow S^2, \quad N \longrightarrow \infty,$$

where $S^2 = \sigma^2 Var(R_k)$. Because we will estimate the variance of Y_k as

$$D_Y \equiv \widehat{Var(Y_k)} = \frac{1}{N}\sum_{k=1}^{N}(X_k - X_{k-1})^2,$$

we will estimate the limit S^2 by the expression

$$\hat{\mathbf{S}}_N^2 = \frac{1}{N} D_Y = \frac{1}{N^2}\sum_{k=1}^{N}(X_k - X_{k-1})^2. \tag{5}$$

Let us define the sequence Z_k as $Z_k = X_k - X_{k-2},\ k = 2, \ldots, N$. We can easily verify that it satisfies $E(Z_k) = 0$ and $Cov(Z_k, Z_{k+1}) = Var(R_k)$. If we substitute the first covariance of the sequence Z_k with its empirical value, we get the estimate of $Var(R_k)$ as

$$\hat{\mathbf{V}}_N^2 \equiv \widehat{Var(R_k)} = \frac{1}{N-2}\sum_{k=2}^{N-1}(X_{k+1} - X_{k-1})(X_k - X_{k-2}), \tag{6}$$

and, according to this and (5), we obtain the estimate of the noise variance

$$\hat{\sigma}_N^2 = \frac{\hat{\mathbf{S}}_N^2}{\hat{\mathbf{V}}_N^2} = \frac{(N-2)\sum\limits_{k=1}^{N}(X_k - X_{k-1})^2}{N^2 \sum\limits_{k=2}^{N-1}(X_{k+1} - X_{k-1})(X_k - X_{k-2})}. \tag{7}$$

Finally, the estimate of the parameter a will be obtained according to (4), (6) and (7). If

$$\hat{a}_N = \frac{-1 + \sqrt{1 + 12\,\hat{\mathbf{V}}_N^2\big(\hat{\sigma}_N^2 + 1\big)^{-1}}}{2}, \tag{8}$$

then the greatest value of the random variable η_k will be $\hat{a} = \min\{[\hat{a}_N] + 1, [\lambda_0]\}$.

3. Simulation and application of the model

We simulated the D-ACD model by Monte Carlo simulation. Figure 2 shows histograms of the simulated values based on 40 independent Monte Carlo simulations of $N = 500$ trails. As it was done in Stojanović [5], we used Poisson's normalized distribution for ε_k defined as

$$P\big\{\varepsilon_k = m\big\} = \frac{1}{e \cdot m!}, \quad m \geq 0, \quad k \geq 1. \tag{9}$$

On the other hand, we suppose that $A = \{-2, -1, 0, 1, 2\}$ and all the random variables of the sequence η_k are uniformly distributed on A. Therefore, the true value of two dimensional parameter is $(a, \sigma^2) = (2, 1)$.

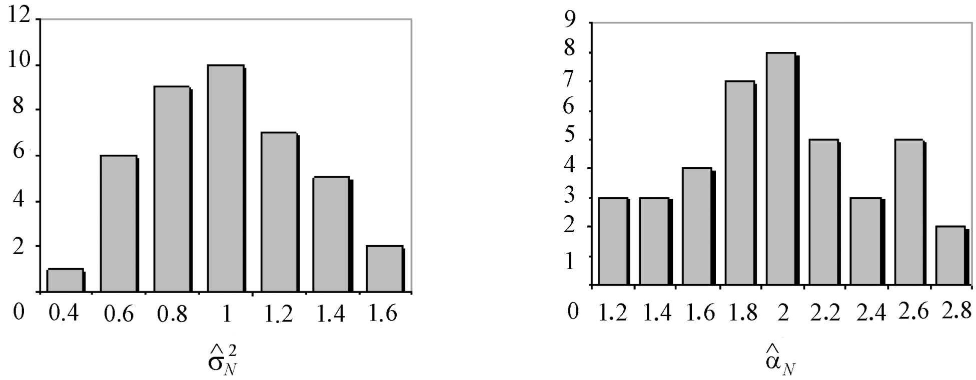

Fig. 2. Histograms of the empirical distributions of the estimated parameters

Under these assumptions, it is obvious that there exists the tendency of cumulation around true values, i.e. of asymptotically normal distribution of the estimated values of two parameters. Naturally, this tendency is more emphatic in the case of $\hat{\sigma}_N^2$. The estimate of a is calculated using $\hat{\sigma}_N^2$, so, its asymptotic behavior depends on $\hat{\sigma}_N^2$.

Also, we applied the described procedure on the official data set from Belgrade Stock Exchange. We considered some of the companies which stocks are traded there in the period when their prices had been registered daily. The estimated data of some leading Serbian companies are shown in Table.

As we can see, the biggest value of the limit a has the company "DIN" from Niš. That indicates that the price of its stocks changes vary slowly, the slowest of all stocks that we considered. On the other hand, it is interesting to consider "Hemofarm" and "Metalac" because they have the estimated values of noise variance $\sigma^2 \approx 0$. It means that the random variables of the sequence ε_k are distributed as the constant equal 1 almost sure. Then it is,

$$X_k \equiv \lambda_k \stackrel{as}{=} \lambda_0 + \sum_{j=0}^{k-1} \eta_j, \quad \forall k \geq 1,$$

and the members of the sequence $(X_k - \lambda_0)$ represents the sum of independent random variables which are uniformly distributed on A.

Estimated values of parameters

Parameters	Companies				
	Alfa Plam Vranje	**DIN** Niš	**Hemofarm** Vršac	**Metalac** G. Milanovac	**Sunce** Sombor
$\hat{\lambda}_0$	3.045	10.90	1.724	2.489	3.675
D_Y	28.09	130.8	3.232	13.48	43.37
$\hat{S}_N^2$	1.221	4.088	0.108	0.139	1.058
$\hat{V}_N^2$	1.673	10.72	0.982	1.536	1.425
$\hat{\sigma}_N^2$	0.730	0.381	0.110	0.090	0.742
$\hat{a}_N$	1.275	4.352	1.204	1.616	2.144
$\hat{a}$	2	5	1	2	3

The other extreme case, when $\hat{a}_N \approx 0$, indicates the hypothesis that $\eta_k \stackrel{a.s.}{=} 0$ and, according to (2),

$$\lambda_k \stackrel{a.s.}{=} \lambda_0 \ (= \text{const})$$

for any $k \geq 1$. In that case, X_k will be the i.i.d. sequence and some of other models will suit better, for example D-AST model, introduced by Stojanovič and Popovič [6]. Finally, the stopping time sequence τ_k has the pure regular dynamics if $a \equiv 0$ and $\lambda_0 \equiv 1$.

The D-ACD model defined here, represents the stopping time sequence τ_k. It has the discrete innovations of special type. The model is efficient, specially, when changes in financial index (here price) is low. We can remark that it is convenient to estimate prices when their changes are not regular, i.e. there are no changes in price in several successive days for random number of days.

References

[1] Ширяев А.Н. Основы стохастической финансовой математики. М.: Фазис, 1998.

[2] Engle R.F., Russel J.R. Autoregressive conditional duration: A new model for irregularly spaced transaction data // Econometrica. 1998. Vol. 66. P. 1127–1162.

[3] Bauwens L., Giot P. The logarithmic ACD model: An application to the bid-ask quote process of three nyse stocks // Ann. Économie Statistique. 2000. Vol. 60. P. 117–149.

[4] Meitz M., Teräsvirta T. Evaluating models of autoregressive conditional duration // J. Business and Economic Statistics. 2006. Vol. 24. P. 104–124.

[5] Stojanovič V. P-ACD model of the financial sequences dynamics in stopping time (in Serbian) // Proc. of the Conf. SYM-OP-IS 2005. Vrnjačka Banja, 2005. P. 501–504.

[6] Stojanovič V., Popovič B. Discrete autoregressive stopping time model // Facta Universitatis. Ser. Math. Inform. 2008. Vol. 23. P. 49—62.

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