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FEM: linear advection and Hermite elements. Part I. Systems of the difference schemes

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The author dedicates this paper to Academician Yurii Ivanovich Shokin who over the past fifty five years has developed the theory of the modified partial differential equation (MDE) for the constant-wind-speed advection equation, advection-diffusion equation and for scalar hyperbolic conservation laws. Shokin, Yanenko and Vorozhtsov present a very precise approach to this problem. These scientists call the modified partial differential equation: Γ- and Π-form of the first differential approximation or Γ- and Π-form of the differential representation of the difference scheme. In the case of hyperbolic equations, the Π-form indicates the value of the Courant number of the separated dissipative or dispersive difference schemes. It also leads to the definition criterion of the stability of the difference method. In the monograph Shokin, Yanenko and Vorozhtsov stability criteria and examples of the abbreviated forms of MDE difference schemes, most popular in the mid-1980s are presernted in the tabular form.

Keywords: finite element method, finite difference method, Hermite elements, Thomas's algorithm.

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Introduction

The linear advection equation is solved in a different way, i. e. by applying Hermite approximations based on piecewise quadratic, cubic and fifth degree polynomials with small compact support. This also means that the initial-boundary problems can be solved in a different way than the very well known way used in the numerical approximation by finite element methods with Lagrange elements or numerical approximation by finite difference methods. This is not a theoretical paper. It presents some new numerical methods for solving partial differential equations. In authors' opinion it could be an inspiration for the theoretical mathematicians to give a precise description of finite element methods with Hermite approximations for solving partial differential equations and their systems of parabolic and hyperbolic types. In our opinion these solutions are better than those based on finite difference methods and finite element methods with Lagrange elements. The fundamentals of the theory of finite element methods (FEM) were initially developed by Zienkiewicz, Taylor, Strang and Fix in the late 1960s and early 1970s [1–8]. In the following years, some authors [9–11] discussed mainly the mathematical theory of FEM (Brenner and Scott [12], Di Pietro and Ern [13]; Ern and

Guermond [14–17]; Samarskii and Andreev [18], Marchuk and Agoshkov [19]; Demkowicz and Gopalakrishnan [20] or Larsson and Thomée [21]) but they did not present any solutions of the initial-value problems. Many numerical experiments can be found in the monographs of Zienkiewicz, Taylor, Norrie and de Vries [1, 2, 22, 23] (mainly for solving the engineering problems and for engineering applications) and by Fletcher [24] and Lynch [25] (for applying FEM to ordinary and partial differential equations).

The first discussion on piecewise Hermite interpolation with applications to partial differential equations is included in [26]. Then Strang and Fix [8] derived a difference scheme for elliptic equations using Hermite cubic elements, and Durran [27] applied them to derive a difference scheme for the advection equation with constant wind speed. A brief analysis of the truncation error and the order of accuracy of the resulting method can also be found in [27, p. 227]. However, the authors did not present the results of numerical experiments.

The theory of solving partial differential equations with Hermite elements is not very popular. It is worth to notice that Durran presented a brief information to the cubic Hermite expansion function in the first edition of his monograph [27, Sect. 4.5.4] but this fragment is not included in the second edition [28].

More popular both in theory and in practice, and in widely published numerical experiments are approximations with piecewise linear chapeau functions [25, 29–31] and piecewise quadratic functions called Lagrange elements [25, 30, 32, 33].

The main theorems introducing the Hermite finite element theory are recalled in Sect. 1. Difference schemes for the advection equation with the piecewise quadratic, cubic and quintic Hermite elements are derived in Sect. 2. The initial and boundary conditions for all nodal parameters and numerical experiments will be discussed in Sect. 2 in Part II of this paper [1]. The basis functions, the piecewise quintic Hermite polynomials, are presented in Appendix A. Our analysis follows Strang and Fix and expands it to quintic polynomials. The terminology: piecewise quadratic and cubic is taken from the monographs of Strang and Fix [8, 11, 18, 34]. The piecewise quintic polynomial is a proposal of the author. Quintic polynomials have not been published before.

1. The basic theorems

Let $\mathcal{H}_c^s$ denote the space of functions with compact support whose derivatives of the order $q \le s$ lie in L^2. The following norms are associated with $\mathcal{H}_c^s$:

$$||e^h||^2_{\mathcal{H}^s} = \sum_{|\alpha|\le s} ||D^\alpha e^h||^2_{L^2}, \qquad ||e^h||^2_{W^s_\infty} = \sum_{|\alpha|\le s} ||D^\alpha e^h||_{L^\infty}$$

which measure of error:

$$e^h = u^h - u$$

and its derivatives of the order $|\alpha| = \sum \alpha_i \le s$:

$$D^\alpha e^h = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} e^h$$

where u^h — approximate solution, u — exact solution, and $\mathcal{H}^s$ is Sobolev W_2^s space.

[1] Winnicki I. FEM: linear advection and Hermite elements. Part II. Numerical experiments. Computational Technologies. (In Press)

In our discussion the derivatives are continuous. Let us consider the following theorems proved in 1970 by Strang and Fix [5, 26].

Theorem 1. *(Strang, Fix) Suppose $\hat{\varphi}(x)$ is in $\mathcal{H}_c^p$. Then the following conditions are equivalent:*

(i) *$\hat{\varphi}(0) \neq 0$, but $\hat{\varphi}(x)$ has zeros of order at least $p+1$ at the other points of $2\pi\mathbb{Z}^n$:*

$$D^\alpha \hat{\varphi}(2\pi j) = 0 \quad \text{if } j \in \mathbb{Z}^n \backslash \{0\},\ |\alpha| \leq p;$$

(ii) *for $|\alpha| \leq p$, $\sum\limits_{j\in\mathbb{Z}^n} j^\alpha \hat{\varphi}(t-j)$ is a polynomial in $t_1, \ldots, t_n$ with leading term Ct^α, $C \neq 0$;*

(iii) *for each $u(x)$ in $\mathcal{H}_c^{p+1}$ there are weights w_j^h such that as $h \to 0$,*

$$\left\| u - \sum w_j^h \varphi_j^h \right\|_{\mathcal{H}^s} \leq c_s h^{p+1-s} ||u||_{\mathcal{H}^{p+1}}, \qquad 0 \leq s \leq p, \qquad \sum |w_j^h|^2 \leq K ||u||^2_{\mathcal{H}^0}.$$

The constants c_s and K are independent of u, $\hat{\varphi}$ — denotes the Fourier transform of φ and $\mathbb{Z}^n$ — set of the integer numbers. Where φ_j^h — non-zero functions within only a finite number of elements [5, p. 816].

Theorem 2. *(Strang, Fix) Suppose $\varphi_1(x), \ldots, \varphi_N(x)$ are in $\mathcal{H}_c^q$. Then the following conditions are equivalent:*

(i) *there are linear combinations $\psi_\alpha(x)$ of the $\varphi_i(x)$ which satisfy*

$$\hat{\psi}_0(0) = 1, \qquad \hat{\psi}_0(2\pi j) = 0 \quad \text{for } j \in \mathbb{Z}^n \backslash \{0\},$$

$$\sum_{\beta \leq \alpha} \frac{D^\beta \hat{\psi}_{\alpha-\beta}(2\pi j)}{\beta! i^{|\beta|}} = 0 \quad \text{for all } j \in \mathbb{Z}^n,\ 1 \leq |\alpha| \leq p;$$

(ii) *there are linear combinations $\psi_\alpha(x)$ of the $\varphi_i(x)$ which satisfy*

$$\frac{t^\alpha}{\alpha!} = \sum_{\beta \leq \alpha} \sum_j \frac{j^\beta \psi_{\alpha-\beta}(t-j)}{\beta!} \quad \text{for } |\alpha| \leq p;$$

(iii) *for each $u(x) \in \mathcal{H}^{p+1}$, there are weights $w_{i,j}^h$ such that for $s = 0, 1, \ldots, q$,*

$$\left\| u - \sum w_{i,j}^h \varphi_{i,j}^h \right\|_{\mathcal{H}^s} \leq c_s h^{p+1-s} ||u||_{\mathcal{H}^{p+1}}, \qquad \sum_{i,j} |w_{i,j}^h|^2 \leq K ||u||^2_{\mathcal{H}^0},$$

where $\varphi_{i,j}^h$ — the basis that the Rayleigh – Ritz – Galerkin principle will select an approximation u^h (1) *[5, p. 799]; $|\alpha| = \alpha_1 + \ldots + \alpha_n$; $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\alpha! = \alpha_1! \cdots \alpha_n!$.*

Theorems 1 and 2 are the basis of algorithms for constructing of Hermite basis functions with effective interpolation properties and then for the difference schemes associated with them.

The technique with piecewise Hermite interpolation by third degree polynomials was first published in 1973 for the stationary equation [8, 11]:

$$-(pu')' + qu = f. \tag{1}$$

The Hermite difference equations for (1) depend on the values of displacements (or heights) $[u_{i-1}, u_i, u_{i+1}]$ and slopes $[u'_{i-1}, u'_i, u'_{i+1}]$. This means that the FEM with Hrmite type elements leads to a system of difference schemes.

Let us consider the 1-D initial-boundary value problem [7, 18, 24, 33–41]:

$$\frac{\partial \mathbf{U}}{\partial t} + L\mathbf{U} = \mathbf{F}, \tag{2}$$

where: L — the first order time-independent operator, $\mathbf{U}$ is the vector of unknowns and $\mathbf{F}$ is the vector of the right-hand side. We also assume that for (2) one can define the corresponding initial and boundary conditions. In geophysical fluid dynamics the shallow water or gravity waves equations are the best examples for the system (2) and for the vector $\mathbf{U}$ [27, 28, 42].

The initial condition (4) describes the propagation of the jump to the right at a constant speed $c = 1$ and is the main condition for testing the majority of the property of the difference method: errors of phase, dispersion (the decay or the growth of the sawtooth type oscillations appearing in the solutions) and dissipation [43–47]. It served the initial condition for modelling explicit schemes, such as: Lax – Wendroff [48–50], upstream, leapfrog [27, 51], Godunov, Crowley [52, 53], Zalesak [27, 28], Beam – Warming [54, 55], Lax – Friedrichs [27] or MacCormack [48, 56], and implicit ones: classical Crank – Nicolson mixed linear and quadratic finite-element Galerkin – Crank – Nicolson type, tridiagonal compact Lele scheme of the fourth and sixth order [27, 28] and others. The initial condition (4) can also be applied to the little-known explicit schemes of Landau [57, p. 430] and Wendroff [21, p. 196]. It should also be noticed that for linear advection the MacCormack and Lax – Wendroff difference schemes have the same form. An exact analysis of both schemes for a scalar hyperbolic conservation law and an irregular initial condition of the right-moving shock wave type is presented in [48].

Linear advection equation for constant wind speed is also a good simple example:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{3}$$

which is consistent with the classic initial condition in the form:

$$u(x,0) = u^0(x) = \begin{cases} u_1 & x \le x_0, \\ u_2 & x > x_0, \end{cases} \qquad u_1 > u_2. \tag{4}$$

In Sect. 2 difference schemes are derived for the initial-value problem (3), (4) assuming that the solution $u_h(x,t)$ approaches by the following series:

1. For the Hermite piecewise quadratic ($d = s = 2$) and piecewise cubic elements ($d = s = 3$):

$$u_h(x,t) = \sum_{i=1}^{N-1} u_i(t)\varphi_i^{(d)}(x) + \sum_{i=1}^{N-1} u_i'(t)\psi_i^{(s)}(x), \tag{5}$$

$$u_h(x,0) = \sum_{i=1}^{N-1} u_{i(d)}^0(0)\varphi_i^{(d)}(x) + \sum_{i=1}^{N-1} u_{i(s)}'^0(0)\psi_i^{(s)}(x) \tag{6}$$

and then $\mathbf{U} = [u_i(t), u_i'(t)]^T$.

2. For the Hermite piecewise fifth degree elements ($d = s = c = 5$; d, s, c — the elements degree):

$$u_h(x,t) = \sum_{i=1}^{N-1} u_i(t)\varphi_i^{(d)}(x) + \sum_{i=1}^{N-1} u_i'(t)\psi_i^{(s)}(x) + \sum_{i=1}^{N-1} u_i''(t)\gamma_i^{(c)}(x), \tag{7}$$

$$u_h(x,0) = \sum_{i=1}^{N-1} u_{i(d)}^0(0)\varphi_i^{(d)}(x) + \sum_{i=1}^{N-1} u_{i(s)}'^0(0)\psi_i^{(s)}(x) + \sum_{i=1}^{N-1} u_{i(c)}''^0(0)\gamma_i^{(c)}(x) \tag{8}$$

and then $\mathbf{U} = [u_i(t), u_i'(t), u_i''(t)]^T$.

This means that in this paper, approximations (5), (6) are applied twice, while approximations (7), (8) are applied only once. Approximations (5)–(8) lead to systems of implicit difference schemes which always extend to only three mesh points: $[i-1, i, i+1]$. The functions $\varphi_i(x)$ with the superscript (d) correspond to the displacement of the solution $u_i(t)$; the functions $\psi(x_i)$ with the superscript (s) correspond to the slopes $u_i'(t)$ of the solution and the functions $\gamma_i(x)$ with the superscript (c) correspond to the curvatures $u_i''(t)$ of the solution. The difference schemes derived from (5) and (6) relate both the displacements and the slopes $\left[u_{i-1}, u_{i-1}', u_i, u_i', u_{i+1}, u_{i+1}'\right]$ as unknowns, while the difference schemes derived from (7) and (8) relate both the displacements and the slopes with the curvatures (or bends) $\left[u_{i-1}, u_{i-1}', u_{i-1}'', u_i, u_i', u_i'', u_{i+1}, u_{i+1}', u_{i+1}''\right]$ as unknowns. Thus, finite element difference equations are systems of two equations in the case of (5) and (6) and a system of three equations in the case of (7) and (8). Despite the increase in the number of unknowns, as already mentioned above, the schemes still extend to only three mesh points: $[i-1, i, i+1]$.

2. The difference schemes in the Hermitian FEM spaces

FEM for advection or diffusion equations with Hermite elements always leads to systems of implicit difference equations (see their forms presented below).

2.1. The Hermite piecewise-quadratic elements

In this case the series (5) and (6) take the form of:

$$u_h(x,t)=\sum_{i=1}^{N-1} u_i(t)\varphi_i^{(2)}(x)+\sum_{i=1}^{N-1} u_i'(t)\psi_i^{(2)}(x), \quad u_h(x,0)=\sum_{i=1}^{N-1} u_{i(2)}^0(0)\varphi_i^{(2)}(x)+\sum_{i=1}^{N-1} u_{i(2)}'^0(0)\psi_i^{(2)}(x).$$

The formulas for the Hermite elements $\varphi_i^{(2)}(x)$ and $\psi_i^{(2)}(x)$ can be found in [18, p. 84]. Their graphs are presented in Appendix A.

The piecewise quadratic Hermite approximation is given by a pair of implicit one-sided approximation in the time difference equations:

$$\begin{cases} H_{2i}^{n+1} + \lambda L_{2i}^{n+1} = H_{2i}^n, \\ H_{2i}'^{n+1} + \lambda L_{2i}'^{n+1} = H_{2i}'^n \end{cases} \tag{9}$$

where: $\lambda = c\tau/h$ — the Courant number, h — spatial step, τ — time step, c — constant wind speed and:

$$\begin{aligned} H_{2i} &= 56(u_{i-1} + u_{i+1}) + 368u_i + 13h(u_{i-1}' - u_{i+1}'), \\ H_{2i}' &= -13(u_{i-1} - u_{i+1}) - 3h(u_{i-1}' + u_{i+1}') + 10hu_i', \\ L_{2i} &= 240(u_{i+1} - u_{i-1}) - 50h(u_{i-1}' - 2u_i' + u_{i+1}'), \\ L_{2i}' &= 50(u_{i-1} - 2u_i + u_{i+1}) + 10h(u_{i-1}' - u_{i+1}'). \end{aligned}$$

Finally, the difference system (9) takes the extended implicit form:

$$\begin{cases} 56(u_{i-1}^{n+1} + u_{i+1}^{n+1}) + 368u_i^{n+1} + 13h(u_{i-1}'^{n+1} - u_{i+1}'^{n+1}) + \\ +\lambda(240(u_{i+1}^{n+1} - u_{i-1}^{n+1}) - 50h(u_{i-1}'^{n+1} - 2u_i'^{n+1} + u_{i+1}'^{n+1})) = \\ = 56(u_{i-1}^n + u_{i+1}^n) + 368u_i^n + 13h(u_{i-1}'^n - u_{i+1}'^n), \\ -13(u_{i-1}^{n+1} - u_{i+1}^{n+1}) - 3h(u_{i-1}'^{n+1} + u_{i+1}'^{n+1}) + 10hu_i'^{n+1} + \\ +\lambda(50(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}) + 10h(u_{i-1}'^{n+1} - u_{i+1}'^{n+1})) = \\ = -13(u_{i-1}^n - u_{i+1}^n) - 3h(u_{i-1}'^n + u_{i+1}'^n) + 10hu_i'^n. \end{cases} \tag{10}$$

The initial conditions $\mathbf{U}_i^0 = [u_{i(2)}^0, u_{i(2)}'^0]^T$ for (10) are determined on the base of the projections:

$$\begin{cases} H_{(2)i}^0 = \dfrac{480}{h} \displaystyle\sum_{i=1}^{N-1} \int\limits_{x_{i-1}}^{x_{i+1}} (u_{i(2)}^0(x)\varphi_i^{(2)}(x) + u_{i(2)}'^0 \psi_i^{(2)}(x))\varphi_k^{(2)}(x)d, \\ H_{2i}'^0 = \dfrac{480}{h^2} \displaystyle\sum_{i=1}^{N-1} \int\limits_{x_{i-1}}^{x_{i+1}} (u_{i(2)}^0(x)\varphi_i^{(2)}(x) + u_{i(2)}'^0 \psi_i^{(2)}(x))\psi_k^{(2)}(x)dx, \end{cases} \qquad k = 1, \ldots, N-1. \tag{11}$$

FEM for hyperbolic and parabolic problems with linear Lagrange elements leads to implicit tridiagonal difference schemes of the Galerkin – Crank – Nicolson type and with Lagrange quadratic elements to pentadiagonal ones [25, 29].

Various variants of the Thomas algorithm [58] have been used to solve such difference equations. Their forms are presented in [59–67]. In subsequent years this algorithm was extended for three-point vector equations. The best examples of such systems are difference schemes with the Hermite elements in 1-D.

Let us return to the initial-boundary value problem with $c = 1$:

$$\frac{\partial \mathbf{U}}{\partial t} + L\mathbf{U} = 0, \qquad L = \frac{\partial}{\partial x}. \tag{12}$$

The difference system (10) may be rewritten:

$$-\mathbf{P}_2\mathbf{U}_{i-1}^{n+1} + \mathbf{Q}_2\mathbf{U}_i^{n+1} - \mathbf{R}_2\mathbf{U}_{i+1}^{n+1} = \mathbf{F}_i^n \tag{13}$$

where: $\mathbf{U}_i = (u_i, u_i')^T$, $\mathbf{F}_i = (H_{2i}, H_{2i}')^T$ and

$$\mathbf{P}_2 = \begin{bmatrix} -(56 - 240\lambda) & h(50\lambda - 13) \\ -(-13 + 50\lambda) & h(3 - 10\lambda) \end{bmatrix}, \qquad \mathbf{Q}_2 = \begin{bmatrix} 368 & 100\lambda h \\ -100\lambda & 10h \end{bmatrix},$$

$$\mathbf{R}_2 = \begin{bmatrix} -(56 + 240\lambda) & h(13 + 50\lambda) \\ -(13 + 50\lambda) & h(3 + 10\lambda) \end{bmatrix}.$$

The procedure of solving the unsteady initial value problem of the (12) type:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad \text{or} \quad \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

using the Hermite elements leads to the system (13) which may be presented in the following Thomas form:

$$\begin{array}{ll} \mathbf{C}_0\mathbf{U}_0 - \mathbf{B}_0\mathbf{U}_1 = \mathbf{F}_0, & i = 0, \\ -\mathbf{A}_i\mathbf{U}_{i-1} + \mathbf{C}_i\mathbf{U}_i - \mathbf{B}_i\mathbf{U}_{i+1} = \mathbf{F}_i, & 1 \le i \le N-1, \\ -\mathbf{A}_N\mathbf{U}_N + \mathbf{C}_N\mathbf{U}_N = \mathbf{F}_N, & i = N \end{array} \tag{14}$$

where: $\mathbf{U}_i$ and $\mathbf{F}_i$ are vectors and $\mathbf{A}_i$, $\mathbf{C}_i$, $\mathbf{B}_i$ are 2×2 matrices and they correspond to $\mathbf{P}_i$, $\mathbf{Q}_i$, $\mathbf{R}_i$, respectively. To solve the block-tridiagonal system of equations (14) we apply the method of elimination three-point vector equations similar to the method of eliminating three-point scalar equations (see: [63] as well as [24, App. A–C]). We will seek the solution (14) in the following form:

$$\mathbf{U}_i = \mathbf{X}_{i+1}\mathbf{U}_{i+1} + \mathbf{Y}_{i+1}, \qquad i = N-1, N-2, \ldots, 0$$

where: $\mathbf{X}_{i+1}$ — the set of the matrices of size 2×2, and $\mathbf{Y}_{i+1}$ — the set of the vectors of dimension 2. Below are the well-known recurrence relations for calculating $\mathbf{X}_{i+1}$ and $\mathbf{Y}_{i+1}$:

$$\begin{array}{lll} \mathbf{X}_{i+1} = (\mathbf{C}_i - \mathbf{A}_i\mathbf{X}_i)^{-1}\mathbf{B}_i, & i = 1, 2, \dots, N-1, & \mathbf{X}_1 = \mathbf{C}_0^{-1}\mathbf{B}_0, \quad (a) \\ \mathbf{Y}_{i+1} = (\mathbf{C}_i - \mathbf{A}_i\mathbf{X}_i)^{-1}(\mathbf{F}_i + \mathbf{A}_i\mathbf{Y}_i), & i = 1, 2, \dots, N, & \mathbf{Y}_1 = \mathbf{C}_0^{-1}\mathbf{F}_0, \quad (b) \\ \mathbf{U}_i = \mathbf{X}_{i+1}\mathbf{U}_{i+1} + \mathbf{Y}_{i+1}, & i = N-1, N-2, \dots, 0, & \mathbf{U}_N = \mathbf{Y}_{N+1}. \quad (c) \end{array} \tag{15}$$

We say that (15) is stable if $||\mathbf{X}_i|| \le 1$ for $1 \le i \le N$.

2.2. The Hermite piecewise-cubic elements

The formulas of the Hermite elements $\varphi_i^{(3)}(x)$ and $\psi_i^{(3)}(x)$ can be found in [11, 18, 19, 27]. Their graphs are presented in Appendix A. Repeating the procedure described in Subsect. 2.1, we obtain:

$$\begin{cases} H_{3i}^{n+1} + \lambda L_{3i}^{n+1} = H_{3i}^{n}, \\ H_{3i}'^{n+1} + \lambda L_{3i}'^{n+1} = H_{3i}'^{n} \end{cases} \tag{16}$$

where:

$$\begin{aligned} H_{3i} &= 54(u_{i-1} + u_{i+1}) + 312u_i + 13h(u'_{i-1} - u'_{i+1}), \\ H'_{3i} &= -13(u_{i-1} - u_{i+1}) - 3h(u'_{i-1} + u'_{i+1}) + 8hu'_i, \\ L_{3i} &= 210(u_{i+1} - u_{i-1}) - 42h(u'_{i-1} - 2u'_i + u'_{i+1}), \\ L'_{3i} &= 42(u_{i-1} - 2u_i + u_{i+1}) + 7h(u'_{i-1} - u'_{i+1}). \end{aligned}$$

Implicit difference schemes for piecewise cubic Hermite elements have a form

$$\begin{cases} 54(u_{i-1}^{n+1} + u_{i+1}^{n+1}) + 312u_i^{n+1} + 13h(u_{i-1}'^{n+1} - u_{i+1}'^{n+1}) + \\ +\lambda(210(u_{i+1}^{n+1} - u_{i-1}^{n+1}) - 42h(u_{i-1}'^{n+1} - 2u_i'^{n+1} + u_{i+1}'^{n+1})) = \\ = 54(u_{i-1}^{n} + u_{i+1}^{n}) + 312u_i^{n} + 13h(u_{i-1}'^{n} - u_{i+1}'^{n}), \\ -13(u_{i-1}^{n+1} - u_{i+1}^{n+1}) - 3h(u_{i-1}'^{n+1} + u_{i+1}'^{n+1}) + 8hu_i'^{n+1} + \\ +\lambda(42(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}) + 7h(u_{i-1}'^{n+1} - u_{i+1}'^{n+1})) = \\ = -13(u_{i-1}^{n} - u_{i+1}^{n}) - 3h(u_{i-1}'^{n} + u_{i+1}'^{n}) + 8hu_i'^{n}. \end{cases}$$

The initial conditions $\mathbf{U}_i^0 = [u_{i(3)}^0, u_{i(3)}'^0]^T$ for (2.2) are similar to (11). The abbreviated form of (2.2) is presented below:

$$-\mathbf{P}_3\mathbf{U}_{i-1}^{n+1} + \mathbf{Q}_3\mathbf{U}_i^{n+1} - \mathbf{R}_3\mathbf{U}_{i+1}^{n+1} = \mathbf{F}_i^n$$

where: $\mathbf{U}_i = [u_i, u'_i]^T$, $\mathbf{F}_i = [H_{3i}, H'_{3i}]^T$ and

$$\mathbf{P}_3 = \begin{bmatrix} -(54-210\lambda) & h(42\lambda-13) \\ -(-13+42\lambda) & h(3-7\lambda) \end{bmatrix}, \quad \mathbf{Q}_3 = \begin{bmatrix} 312 & 84\lambda h \\ -84\lambda & 8h \end{bmatrix}, \quad \mathbf{R}_3 = \begin{bmatrix} -(54+210\lambda) & h(13+42\lambda) \\ -(13+42\lambda) & h(3+7\lambda) \end{bmatrix}.$$

2.3. The piecewise fifth degree Hermite elements

The formulas for the elements $\varphi_i^{(5)}(x)$, $\psi_i^{(5)}(x)$ and $\gamma_i^{(5)}(x)$ and their graphs are presented in Appendix A. In this case there is a triple node at each point $x_i = ih$:

$$u_h(x,t) = \sum_{i=1}^{N-1} u_i(t)\varphi_i^{(5)}(x) + \sum_{i=1}^{N-1} u'_i(t)\psi_i^{(5)}(x) + \sum_{i=1}^{N-1} u''_i(t)\gamma_i^{(5)}(x),$$

$$u_h(x,0) = \sum_{i=1}^{N-1} u_{i(5)}^0(0)\varphi_i^{(5)}(x) + \sum_{i=1}^{N-1} u_{i(5)}'^0(0)\psi_i^{(5)}(x) + \sum_{i=1}^{N-1} u_{i(5)}''^0(0)\gamma_i^{(5)}(x).$$

The following system of three difference equations is obtained:

$$\begin{cases} H_{5i}^{n+1} + \lambda L_i^{n+1} = H_{5i}^n, \\ H_{5i}'^{n+1} + \lambda L_i'^{n+1} = H_{5i}'^n, \\ H_{5i}''^{n+1} + \lambda L_i''^{n+1} = H_{5i}''^n \end{cases} \tag{17}$$

where:

$$\begin{aligned} H_{5i} &= 6000(u_{i-1}+u_{i+1}) + 43440u_i + 1812h(u'_{i-1}-u'_{i+1}) + 181h^2(u''_{i-1}+u''_{i+1}) + 562h^2u''_i, \\ H'_{5i} &= -1812(u_{i-1} - u_{i+1}) - 532h(u'_{i-1} + u'_{i+1}) + 1664hu'_i - 52h^2(u''_{i-1} - u''_{i+1}), \\ H''_{5i} &= 181(u_{i-1} - u_{i+1}) + 562u_i + 52h(u'_{i-1} - u'_{i+1}) + 5h^2(u''_{i-1} + u''_{i+1}) + 12h^2u''_i, \\ L_{5i} &= 27720(u_{i+1} - u_{i-1}) - 7260h(u'_{i-1} - 2u'_i + u'_{i+1}) - 660h^2(u''_{i-1} - u''_{i+1}), \\ L'_{5i} &= 7260(u_{i-1} - 2u_i + u_{i+1}) + 1716h(u'_{i-1} - u'_{i+1}) + 143h^2(u''_{i-1} + u''_{i+1}) + 110h^2u''_i, \\ L''_{5i} &= 660(u_{i+1} - u_{i-1}) - 143h(u'_{i-1} + u'_{i+1}) - 110hu'_i - 11h^2(u''_{i-1} - u''_{i+1}). \end{aligned}$$

The system (18) for the fifth degree Hermite elements is presented below:

$$-\mathbf{P}_5\mathbf{U}_{i-1}^{n+1} + \mathbf{Q}_5\mathbf{U}_i^{n+1} - \mathbf{R}_5\mathbf{U}_{i+1}^{n+1} = \mathbf{F}_i^n$$

where $\mathbf{U}_i = [u_i, u'_i, u''_i]^T$, $\mathbf{F}_i = [H_{5i}, H'_{5i}, H''_{5i}]^T$ and

$$\mathbf{P}_5 = \begin{bmatrix} -6000 + 27720\lambda & h(-1812 + 7260\lambda) & h^2(-181 + 660\lambda) \\ 1812 - 7260\lambda & h(532 - 1716\lambda) & h^2(52 - 143\lambda) \\ -181 + 660\lambda & h(-52 + 143\lambda) & h^2(-5 + 11\lambda) \end{bmatrix},$$

$$\mathbf{Q}_5 = \begin{bmatrix} 43440 & 14520\lambda h & 562h^2 \\ -14520\lambda & 1664h & 110\lambda h^2 \\ 562 & -110\lambda h & 12h^2 \end{bmatrix},$$

$$\mathbf{R}_5 = \begin{bmatrix} -6000 - 27720\lambda & h(1812 + 7260\lambda) & -h^2(181 + 660\lambda) \\ -1812 - 7260\lambda & h(532 + 1716\lambda) & -h^2(52 + 143\lambda) \\ -181 - 660\lambda & h(52 + 143\lambda) & -h^2(5 + 11\lambda) \end{bmatrix}.$$

The difference schemes for the fifth degree Hermite elements have the form:

$$\begin{cases} 6000(u_{i-1}^{n+1} + u_{i+1}^{n+1}) + 43440u_i^{n+1} + 27720\lambda(u_{i+1}^{n+1} - u_{i-1}^{n+1}) + 1812h(u_{i-1}'^{n+1} - u_{i+1}'^{n+1}) - \\ -7260h\lambda(u_{i-1}'^{n+1} - 2u_i'^{n+1} + u_{i+1}'^{n+1}) + 181h^2(u_{i-1}''^{n+1} + u_{i+1}''^{n+1}) + 562h^2u_i''^{n+1} - \\ -660h^2\lambda(u_{i-1}''^{n+1} - u_{i+1}''^{n+1}) = 6000(u_{i-1}^n + u_{i+1}^n) + 43440u_i^n + 1812h(u_{i-1}'^n - u_{i+1}'^n) + \\ +181h^2(u_{i-1}''^n + u_{i+1}''^n) + 562h^2u_i''^n, \\ -1812(u_{i-1}^{n+1} - u_{i+1}^{n+1}) + 7260\lambda(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}) - 532h(u_{i-1}'^{n+1} + u_{i+1}'^{n+1}) + \\ +1664hu_i'^{n+1} + 1716h\lambda(u_{i-1}'^{n+1} - u_{i+1}'^{n+1}) - 52h^2(u_{i-1}''^{n+1} - u_{i+1}''^{n+1}) + \\ +143h^2\lambda(u_{i-1}''^{n+1} + u_{i+1}''^{n+1}) + 110h^2\lambda u_i''^{n+1} = -1812(u_{i-1}^n - u_{i+1}^n) - \\ -532h(u_{i-1}'^n + u_{i+1}'^n) + 1664hu_i'^n - 52h^2(u_{i-1}''^n - u_{i+1}''^n), \\ 181(u_{i-1}^{n+1} - u_{i+1}^{n+1}) + 562u_i^{n+1} + 660\lambda(u_{i+1}^{n+1} - u_{i-1}^{n+1}) + 52h(u_{i-1}'^{n+1} - u_{i+1}'^{n+1}) - \\ -143h\lambda(u_{i-1}'^{n+1} + u_{i+1}'^{n+1}) - 110h\lambda u_i'^{n+1} + 5h^2(u_{i-1}''^{n+1} + u_{i+1}''^{n+1}) + 12h^2u_i''^{n+1} - \\ -11h^2\lambda(u_{i-1}''^{n+1} - u_{i+1}''^{n+1}) = 181(u_{i-1}^n - u_{i+1}^n) + 562u_i^n + 52h(u_{i-1}'^n - u_{i+1}'^n) + \\ +5h^2(u_{i-1}''^n + u_{i+1}''^n) + 12h^2u_i''^n. \end{cases} \tag{18}$$

Conclusions

In this paper three systems of difference schemes for the linear advection equation are derived, constructed on the basis of the FEM theory with piecewise quadratic, cubic and quintic Hermite elements. An irregular initial condition described the displacement of a jump moving to the right. To solve these implicit systems (9), (16) and (17) a three-point vector variant of the Thomas algorithm [58] was applied. It is well-known that this algorithm itself can develop oscillations with very different amplitudes. This is especially disadvantageous in the case of irregular initial conditions. They can be generated by recurrence relations for calculating the matrices $\mathbf{X}_{i+1}$ and vectors $\mathbf{Y}_{i+1}$ from $(15),(a)$ and (b). They are determined from the left boundary to the right for $i = 1, 2, \ldots, N-1$ and for $i = 1, 2, \ldots, N$, respectively. And the following relation for calculating the solution vector $\mathbf{U}_i$ from the right boundary to the left for $i = N-1, \ldots, 0$ (see: $(15),(c)$).

The numerical experiment will be presented in the future work[1].

Appendix A: definitions of $\varphi^{(5)}(x_i)$, $\psi^{(5)}(x_i)$, $\gamma^{(5)}(x_i)$ and the graphs of $\varphi^{(2,3,5)}(x_i)$, $\psi^{(2,3,5)}(x_i)$

The Hermite elements of the fifth degree $\varphi_i^{(5)}(x)$, $\psi_i^{(5)}(x)$ and $\gamma_i^{(5)}(x)$ have not been yet published (Fig. 1). Their forms are presented below:

$$\varphi_i^{(5)}(x) = \begin{cases} 6\left(\dfrac{x-x_{i-1}}{h}\right)^5 - 15\left(\dfrac{x-x_{i-1}}{h}\right)^4 + 10\left(\dfrac{x-x_{i-1}}{h}\right)^3, & x_{i-1} \le x \le x_i, \\ -6\left(\dfrac{x-x_{i+1}}{h}\right)^5 - 15\left(\dfrac{x-x_{i+1}}{h}\right)^4 - 10\left(\dfrac{x-x_{i+1}}{h}\right)^3, & x_i \le x \le x_{i+1}, \\ 0, \quad x \notin \langle x_{i-1} \div x_{i+1} \rangle, \end{cases}$$

$$\psi_i^{(5)}(x) = \begin{cases} -3\dfrac{(x-x_{i-1})^5}{h^4} + 7\dfrac{(x-x_{i-1})^4}{h^3} - 4\dfrac{(x-x_{i-1})^3}{h^2}, & x_{i-1} \le x \le x_i, \\ -3\dfrac{(x-x_{i+1})^5}{h^4} - 7\dfrac{(x-x_{i+1})^4}{h^3} - 4\dfrac{(x-x_{i+1})^3}{h^2}, & x_i \le x \le x_{i+1}, \\ 0, \quad x \notin \langle x_{i-1} \div x_{i+1} \rangle, \end{cases}$$

$$\gamma_i^{(5)}(x) = \begin{cases} 0.5\dfrac{(x-x_{i-1})^5}{h^3} - \dfrac{(x-x_{i-1})^4}{h^2} + 0.5\dfrac{(x-x_{i-1})^3}{h}, & x_{i-1} \le x \le x_i, \\ -0.5\dfrac{(x-x_{i+1})^5}{h^3} - \dfrac{(x-x_{i+1})^4}{h^2} - 0.5\dfrac{(x-x_{i+1})^3}{h}, & x_i \le x \le x_{i+1}, \\ 0, \quad x \notin \langle x_{i-1} \div x_{i+1} \rangle. \end{cases}$$

It is easy to check that they satisfy the conditions (19).

$$\begin{cases} \varphi_i^{(5)}(x_i) = 1, & \varphi_i^{(5)}(x_{i-1}) = \varphi_i^{(5)}(x_{i+1}) = 0, \\ & \varphi_i'^{(5)}(x_i) = \varphi_i'^{(5)}(x_{i-1}) = \varphi_i'^{(5)}(x_{i+1}) = 0; \\ \psi_i'^{(5)}(x_i) = 1, & \psi_i'^{(5)}(x_{i-1}) = \psi_i'^{(5)}(x_{i+1}) = 0, \\ & \psi_i^{(5)}(x_i) = \psi_i^{(5)}(x_{i-1}) = \psi_i^{(5)}(x_{i+1}) = 0; \\ \gamma_i''(x_i) = 1, & \gamma_i''(x_{i-1}) = \gamma''(x_{i+1}) = 0, \\ & \gamma_i'(x_i) = \gamma_i'(x_{i-1}) = \gamma_i'(x_{i+1}) = 0, \\ & \gamma_i(x_i) = \gamma_i(x_{i-1}) = \gamma_i(x_{i+1}) = 0. \end{cases} \tag{19}$$

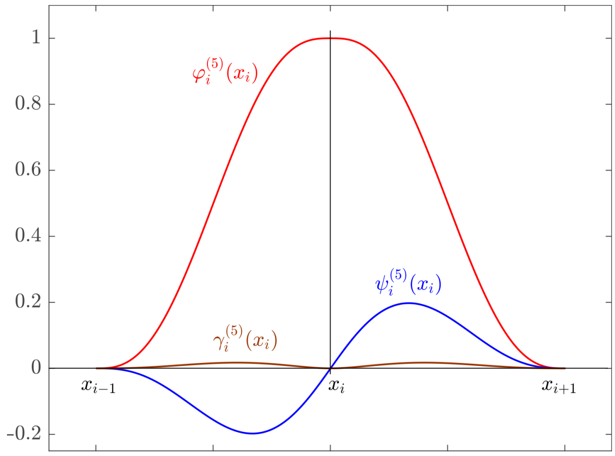

Fig. 1. The graphs of the piecewise quintic basis functions family

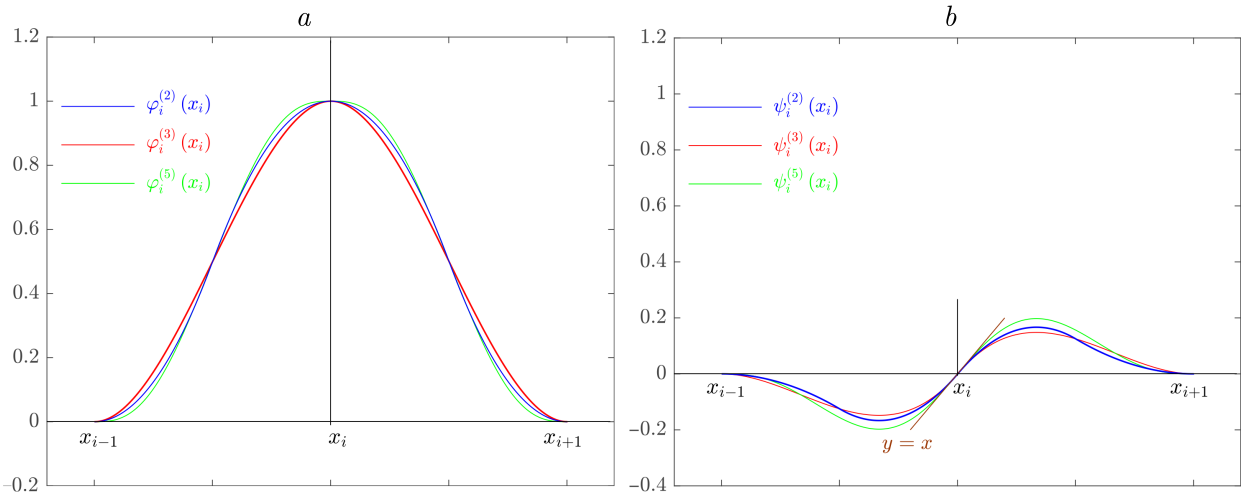

Fig. 2. The graphs of the $\varphi^{(2,3,5)}(x_i)$s (a) and $\psi^{(2,3,5)}(x_i)$s (b) basis functions: the quadratic elements — blue line, the cubic elements — red line and the elements of the fifth degree — green line; the tangent to the $\psi_i^{(2,3,5)}(x)$ for $x = x_i$ — brown line

The graphs of the basis functions $\varphi^{(2,3,5)}(x_i)$s and $\psi^{(2,3,5)}(x_i)$s (Fig. 2) differ slightly. One may even say that it is unnoticeable. But the solutions of $u^{(2,3,5)}(x_i)$ and $u'^{(2,3,5)}(x_i)$ differ very strongly. Especially the values of the second nodal parameters $u'^{(2,3)}(x_i)$ and $u'^{(5)}(x_i)$. Numerical experiments will be discussed in Sect. 2 of Part II of this paper [1].

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ВЫЧИСЛИТЕЛЬНЫЕ ТЕХНОЛОГИИ

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МКЭ: Линейная адвекция и элементы Эрмита. Часть I. Системы разностных схем

И. Винницки

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Аннотация

Автор посвящает эту статью академику Юрию Ивановичу Шокину, который за последние пятьдесят пять лет разработал теорию модифицированного уравнения в частных производных (MDE) для уравнения адвекции с постоянной скоростью ветра, уравнения адвекции-диффузии и для скалярных гиперболических законов сохранения. Шокин, Яненко и Ворожцов представляют очень точный подход к этой проблеме. Эти ученые называют модифицированное уравнение в частных производных Г- и П-формой первого дифференциального приближения или Г- и П-формой дифференциального представления разностной схемы. В случае уравнений гиперболического типа форма П указывает на значение числа Куранта разделенных диссипативных или дисперсионных разностных схем. Она также приводит к определению критерия устойчивости разностного метода. В монографиях Шокина, Яненко и Ворожцова в табличной форме представлены критерии устойчивости и примеры сокращенных форм MDE разностных схем, наиболее популярных в середине 1980-х гг.

Ключевые слова: метод конечных элементов, метод конечных разностей, элементы Эрмита, алгоритм Томаса.

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